1 Propositional Logic

"Why was the elephant wearing your pajamas?"

Propositional logic is the logical[3] place to begin our journey. From philosophical explorations in ancient Greece, formal logic was developed into a field of mathematics and began the first steps towards the computer.

If you've ever gotten in an argument with a friend, then you've used logic before. You've heard people say "let's be **logical** here" or "that makes no (**logical**) sense", but what are we referring to when we use the word "**logic**"?

The basis of logic is comprised of **propositions** and its **operators**, as they are the smallest units of computation we can work with. It also beautifully ties in the calculations that a computer does with 1's and 0's. Let's first get familiar with these fundamental ideas.

1.1 Propositions

When we speak in everyday life, we often make statements that are either **true** or **false**. This type of black and white statements that can be resolved to either **true** or **false** is what we call a **proposition**.

> **Proposition**
> A declarative sentence - a sentence that is either **true** or **false** but not both at the same time.

The following are some examples of propositions:

- The sun is shining today.
- I am the president of the United States.
- 178917248743279764327 is a prime number.
- There are extraterrestrial life outside of Earth.

Notice that each of the above statements can resolve to either **true** or **false**. The sun is either shining today or it is not, but it cannot be both at the same time. "I am the president of the United States" is **false** if I say it, but if the actual president were to say this statement then it would be **true**. 178917248743279764327 is either prime or not, and while it may be difficult to determine, the statement itself will still either be **true** or **false** without question.

The last proposition in the list is a rather odd one. As of current we do not have definitive proof whether of extraterrestrial life. However, regardless of whether or not extraterrestrial life exists, this statement is still a proposition as it must resolve to **true** or **false**.

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[3] Pun intended

Then what is not a proposition? Here are some examples of statements that are not propositions:

- $x + 2 = 4$
- Pay attention in class!
- Is it snowing outside?
- This statement is false.

Let's see why. For the example of $x + 2 = 4$, if I choose x to be 2, then I get $2 + 2 = 4$ which is clearly true. In contrast, if I choose x to be 3, then I get $3 + 2 = 5$ which is not 4, and so the statement is false. There is no definity in the **truth value** of the statement, therefore it is not a proposition. The second statement above is an *imperative* sentence (it's telling you to do something) and the third is a *question*. Hence, none of them are propositions.

What about the forth statement? This is a classic example of a *paradox* — where no matter how try you resolve the statement, you will end up in a never-ending loop of **true** and **false**, thus it cannot be a proposition. Give it a try and see where it takes you!

Exercise 1.1 *Provide one example and one non-example of a proposition.*

1.2 Logical Operators

Let's say we have the following two propositions:

- *"I am a cowboy."*
- *"There is a snake in my boot."*

We would not be too surprised if Woody combines these two propositions and says *"I am a cowboy* ***and*** *there is a snake in my boot"*. This longer statement is still either all **true** or all **false**, and therefore still a proposition. There are quite a few things to break down in this simple statement. First, we differentiate between an **atomic** and a **compound** proposition. Then we will introduce the logical connectors called **logical operators**.

Atomic Proposition
A **proposition** that contains no **logical operators**.

Compound Proposition
A **proposition** that includes the use of one or more **logical operator(s)**.

Logical Operator
An operation that "connects" two[a] logical statements into a compound proposition.

[a] There is a *unary* operator that only modifies a single statement - negation

From the above example, the two propositions *"I am a cowboy"* and *"there is a snake in my boot"* are **atomic propositions** because they are as small of a proposition we can break it down into — they contain no logical operators (we will soon see a comprehensive list of all the basic logical operators to look for). The combined proposition *"I am a cowboy and there is a snake in my boot"* is a **compound proposition** because it combines two **atomic propositions** through use of the **"and" logical operator**.

At this point, we will need a more *formal language* to work with propositions and their logical operators. We can make examples all day long, but we will just get more and more confused. Here, we will begin to develop the notation for **formal logic**.

Propositional Variable
Variables used to refer to **propositions**. The convention is to use lower case (p, q, r, etc.) for **atomic propositions**, and upper case (P, Q, R, etc.) for **compound propositions**.

Let us also first define a set of **logical operators** that are most commonly used, and later look at how these operators alter and combine propositions and their truth values. Here we will also make the distinction between a **unary** and **binary** operator:

Unary Operator
A **logical operator** that alters the truth value of a single **proposition**.

- $\neg$ called **negation**, $\neg p$ is commonly read as *"not p"*.

Binary Operator
A **logical operator** that combines the truth value of two **propositions** into a single truth value.

- $\wedge$ called **"conjunction"**, $p \wedge q$ is commonly read as *"p and q"*.
- $\vee$ called **"disjunction"**, $p \vee q$ is commonly read as *"p or q"*.
- $\rightarrow$ called **"conditional"** or **implication**, $p \rightarrow q$ is read as commonly *"if p then q"*.
- $\longleftrightarrow$ called **"equivalence"** or **"bi-implication"**, $p \longleftrightarrow q$ is commonly read as *"p if and only if q"*.
- $\oplus$ called **"exclusive or"**, $p \oplus q$ is commonly read as *"p or q, but not both"*.

With this set of notation, we can now easily write the compound proposition "I am a cowboy and there is a snake in my boot" in a simpler language. We will first define the following **propositional variables**:

$$p = \text{“I am a cowboy”}$$
$$q = \text{“There is a snake in my boot”}$$

then define the following **compound proposition** using the **conjunction** logical operator:

$$p \wedge q = \text{“I am a cowboy \textbf{and} there is a snake in my boot.”}$$

We can see how much easier it is to navigate logic with **variables** and **logical operators** than it would be to constantly use English examples. In a later section we will get more practice with translating between logic and English.

Notice that our compound proposition $p \wedge q =$ *"I am a cowboy and there is a snake in my boot"* is still a **proposition** — it is either **true** in its entirety or **false** in its entirety, depending on the **truth values** of its **atomic** parts p and q.

1.2.1 Resolving of Logical Operators

Recall that every proposition must be either **true** or **false**, but not both at the same time. This means every proposition and its corresponding variable can be assign a **truth value**.

> **Truth Value**
> The **true** or **false** value of a **proposition**. We use T to denote "**true**" and F to denote "**false**".

Here we will examine how each **logical operators** alter the **truth values** of a propositional statement. The behavior of these **logical operators** can be succinctly summarized in a **truth table**.

> **Truth Table**
> A table illustrating the **truth value** resolutions of every possible combination of input **truth values** for a **logical operator**.

Negation ($\neg$)
Negation is a **unary operator**, meaning it acts on one and only one truth truth value. A negation will change the truth value to the *opposite value*. Given a proposition p:

- If p is **true**, then the negation $\neg p$ will resolve to **false**.
- If p is **false**, then the negation of $\neg p$ will resolve to **true**.

p	$\neg p$
T	F
F	T

Table 1: Truth table for the negation operator.

Conjunction ($\wedge$)
Conjunction is a **binary operator**, meaning it acts upon two truth values. A conjunction will resolve to true when and only when both truth values are **true** at the same time. In short, conjunction is looking for two **true's**. Given two propositions p and q:

- If both p and q are **true**, then the conjunction $p \wedge q$ will resolve to **true**.
- If there is a **false** anywhere between p and q, then the conjunction $p \wedge q$ will resolve to **false**.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 2: Truth table for the conjunction operator.

Disjunction ($\vee$)
Disjunction is a **binary operator**, meaning it acts upon two truth values. A disjunction will resolve to true when and only when there is at least one **true** between the two propositions. In short, disjunction is looking for the presence of a **true** anywhere. Given two propositions p and q:

- If there is a **true** anywhere between p and q, then the disjunction $p \vee q$ will resolve to **true**.
- If both p and q are **false**, then the disjunction $p \vee q$ will resolve to **false**.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 3: Truth table for the disjunction operator.

Implication ($\rightarrow$)
Implication is a **binary operator**, meaning it acts upon two truth values. The implication is a bit more complicated, so we'll need to dissect the anatomy of the operator. An implication statement has two parts: the first proposition, called the **hypothesis**, and the second proposition after the operator, called the **conclusion**.

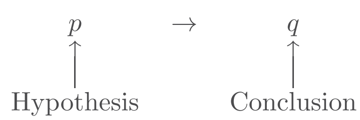

Figure 1: Components of an implication statement

An implication will resolve to true when a **true hypothesis** is followed by a **true conclusion**, or when the hypothesis is **false**[4]. Given two propositions p and q:

- If both the **hypothesis** p and **conclusion** q are **true**, then $p \rightarrow q$ resolves to **true**.
- If the **hypothesis** p is **true** but the **conclusion** q is **false**, then $p \rightarrow q$ resolves to **false**.
- If the **hypothesis** p is **false**, then $p \rightarrow q$ resolves to **true** by default regardless of the truth value of the **conclusion**.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 4: Truth table for the implication operator.

[4]This behavior might seem arbitrary, but we will take a closer look as to whey **implication** behaves as such in the following section when translating between English and Logic in section 1.4

To develop a more intuitive understanding as to why **implication** behaves this way, let's consider the following sentence:

"If an elephant is wearing my pajamas, then I am dreaming."

Let's consider each possible scenario:

Elephant wearing pajamas?	**Am I dreaming?**	**Does this make sense?**	**Truth Value**
T	T	If the elephant is wearing my pajamas, and I am dreaming, then I'm not worried.	T
T	F	If the elephant is wearing my pajamas, but I am not dreaming, then something is wrong.	F
F	T	If the elephant is **not** wearing my pajamas, then I have nothing to worry about.	T
F	F	If the elephant is **not** wearing my pajamas, then I have nothing to worry about.	T

For the first two rows, these make sense. If we start off the scenario with the elephant wearing my pajamas, we want to check if we are dreaming or not.

- The scenario of the elephant wearing my pajamas and I am dreaming makes sense, therefore $T \to T \equiv T$.
- The scenario of the elephant wearing my pajamas and I am not dreaming does not make sense, therefore $T \to F \equiv F$.

The last two rows are more interesting. If the the elephant is not wearing my pajamas (**hypothesis** is **false**), then I can be dreaming, or not dreaming, it doesn't really matter:

- If the elephant is not wearing my pajamas, I can be dreaming for any other number of reasons (a chicken is wearing my pajamas), therefore $F \to T \equiv T$.
- If the elephant is not wearing my pajamas, I can be not dreaming and there is no concern at all, therefore $F \to F \equiv F$.

In summary, when we see an implication $p \to q$, we want to check first if p is **false**. If the hypothesis p is **false**, then we know $p \to q$ is **true** by default and we don't need to check the conclusion q at all. Only when the hypothesis p is **true** do we have to now consider the truth value of the conclusion q.

Bi-implication ($\longleftrightarrow$)
Bi-implication, or **equivalence**, is a **binary operator**, meaning it acts upon two truth values. A bi-implication will resolve to true when and only when both truth values have the exact same truth value (both **true** or both **false** at the same time). In sort, bi-implication is looking for whether the two truth values are identical. Given two propositions p and q:

- If the truth values of p and q are identical to each other, then $p \longleftrightarrow q$ will resolve to **true**.
- If the truth values of p and q are negations of each other, then $p \longleftrightarrow q$ will resolve to **false**

p	q	$p \longleftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 5: Truth table for the bi-implication operator.

Exclusive-or ($\oplus$)
Exclusive-or is a **binary operator**, meaning it acts upon two truth values. A bi-implication will resolve to true when and only when the two truth values are negations of each other. In short, exclusive-or is looking for whether the two truth values are different. Given two propositions p and q:

- If the truth values of p and q are negations of each other, then $p \oplus q$ will resolve to **true**.
- If the truth values of p and q are identical to each other, then $p \oplus q$ will resolve to **false**.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Table 6: Truth table for the bi-implication operator.

As a point of interest, you might notice that the operators **bi-implication** and **exclusive-or** have opposite behavior, which is true. You might also feel that **conjunction** and **disjunction** have opposite behavior, **<u>but this is not the case</u>**. We will take a look at the reasoning to this when we develop a more rigorous approach to analyzing compound propositions in Logical Equivalencies in Section 1.5.

In general, we call these connected propositions variables through use of logical operators **propositional formula**, or just **formula** for short. When we determine the **truth value** of an entire **propositional formula** we call this process **resolving the formula**.

1.2.2 Order of Precedence in Logic

So far we have seen **propositional formulae** containing just two variables and what they resolve to. When we have and formulae with multiple variables we will need a system in which to determine which logical operator to resolve first. Much similar to order of operations you learned in algebra where you would calculate multiplication before addition:

$$\begin{aligned} 3 + 7 \times 6 &= 3 + 42 \\ &= 45 \end{aligned}$$

We also have order of operations for logical formulae. If you are given the following formula:

$$p \to q \wedge r$$

we will need to know if we should resolve the "$\to$" first or resolve the "$\wedge$" first. Similar to *PEMDAS*, we have an **order of precedence** for the logical operators:

Precedence	Operator
1	$()$
2	$\neg$
3	$\wedge$
4	$\oplus$
5	$\vee$
6	$\to$
7	$\longleftrightarrow$

Thus we see that for $p \to q \wedge r$, we should resolve $q \wedge r$ first, then resolve the $p \to$ after.

1.3 Constructing Truth Tables

Now that we know how to resolve logical operators as well as the order of precedence of the operators, we can now resolve more complicated logical formulae and compound propositions. Consider the following worked example:

Example 1.1 *Draw a truth table for* $(p \to q) \vee \neg p$

Since we have two atomic propositions p and q, we want to list out all the possible truth value combinations we can make between p and q. Each row will be a truth value combination we will consider:

p	q	
T	T	
T	F	
F	T	
F	F	

The listing order starting with all T's and flipping each T to F one-by-one until we reach all F's is the conventional order of listing the rows of a truth table. You can technically choose any ordering of listing your rows, but following this convention the a good systematic way to make sure you've covered all possible combinations. Every single worked example will be using this ordering.

Now that we have our truth value combinations set up, let's resolve the first logical operator. Since $p \to q$ is in the parenthesis, we will resolve this first. To do this, we add a column to our truth table for $p \to q$ and resolving it for each truth value combination between p and q.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	F

Next we want to resolve $\neg p$, so we add another column to resolve $\neg p$:

p	q	$p \to q$	$\neg p$
T	T	T	F
T	F	F	F
F	T	T	T
F	F	F	T

And finally, now that we have $p \to q$ and $\neg p$ resolved individually, we can take these two columns and easily resolve the entire formula $(p \to q) \lor \neg p$.

p	q	$p \to q$	$\neg p$	$(p \to q) \lor \neg p$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

As a little bonus, what do you notice about the column for $(p \to q) \lor \neg p$ compared with the other other columns? Do you notice any similarities? What do you think this means? We will take a closer look at what it means when two columns of a truth table are identical in Section[CITE], but for now it might be a fun exercise to ponder this by yourself.

Let's consider a more complicated logical formula to construct a truth table:

Example 1.2 *Draw a truth table for $(p \lor q) \to (q \land r)$*

Here we have three atomic propositions: p, q, and r, meaning we have more combinations of truth values to consider. Let's list them down using the typical conventional ordering:

p	q	r	
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Now we will resolve $p \lor q$ and $q \land r$ individually in separate columns:

p	q	r	$p \lor q$	$q \land r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	F
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

Now with $p \lor q$ and $q \land r$ done as an intermediate step, we can now resolve the full formula $(p \lor q) \rightarrow (q \land r)$ with ease:

p	q	r	$p \lor q$	$q \land r$	$(p \lor q) \rightarrow (q \land r)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	T	F	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	F	T
F	F	F	F	F	T

Exercise 1.2 *Draw the truth table for the following propositional formulae.*

(a) $p \lor \neg p$

(b) $p \land (\neg p \lor \neg p)$

(c) $p \to (\neg q \longleftrightarrow \neg p)$

(f) $p \to ((r \oplus p) \vee \neg q)$

1.3.1 Intermediate Columns

Rather than asking *"how many total columns are there in a truth table,"* it is more efficient to ask *"how many columns (steps) are needed to reach the final column of the formula?"* These working columns are called **intermediate columns**.

> **Intermediate Columns**
> The columns made for each working step towards resolving the final solution column in a **truth table**. The number of **intermediate columns** in a truth table is the sum of the number of negated variables o and number of logical operators p subtracted by 1: $o + p - 1$

Example 1.3 *Determine the number of intermediate columns for the truth table of* $\neg p \wedge \neg(\neg q \vee \neg p) \rightarrow r$

From the previous example, we know there are a total of 9 columns. Three of which are columns for the atomic variables, and one for the final answer column.

Atomic variables			Intermediate Columns					Solution Column
p	q	r	$\neg p$	$\neg q$	$\neg q \vee \neg p$	$\neg(\neg q \vee \neg p)$	$\neg p \wedge \neg(\neg q \vee \neg p)$	$\neg p \wedge \neg(\neg q \vee \neg p) \rightarrow r$

Therefore there are **5 intermediate columns**.

For a more general calculation, there are 2 atomic variables that are negated ($\neg p$ and $\neg q$) along with 4 logical operators (one $\wedge$, one $\neg$ on a compound proposition, one $\vee$, and one $\rightarrow$), therefore requiring a total of $(2 + 4) - 1 = 4$ **intermediate columns**. Recall we subtract 1 since we are overcounting by 1 with the final solution columns.

This process of finding the general case calculation by identifying structural patterns is an important skill to develop in computing as we often faced with calculating complex problem systems. Being able to quickly calculate how many important elements there are in a problem scope helps us get a sense of how the computational complexity required to solve a problem. This is not a skill just in isolation in calculating the number of intermediate columns in a truth table; we get much more practice of this when we reach Unit 2's Graph Theory in Section 5.

Exercise 1.3 *Determine the number of rows and the number of intermediate columns for each of the propositional formulae:*

(a) $p \vee \neg p$

(b) $p \wedge \neg(\neg q \vee \neg p)$

(c) $(p \wedge \neg(\neg q \rightarrow \neg p)) \rightarrow (\neg q \oplus \neg p)$

(f) $p \wedge ((q \vee r) \rightarrow (s \longleftrightarrow t))$

1.3.2 Tautologies and Contradictions

Consider the following truth tables for $p \vee (p \to q)$ and $p \wedge (\neg(q \to p))$:

p	q	$p \to q$	$p \vee (p \to q)$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

p	q	$q \to p$	$\neg(q \to p)$	$p \wedge (\neg(q \to p))$
T	T	T	F	F
T	F	T	F	F
F	T	F	T	F
F	F	T	F	F

you may have noticed there are some compound propositions, regardless of what combination of truth values you assign to the atomic variables, always resolved to true while others always resolved to false. These special cases are called a **tautology** or **contradiction** respectively and lie at the heart of computation and mathematics.

Tautology
A proposition P is a **tautology** if and only if P resolves to **true** for every possible truth value assignment of its variables. In other words, if the truth values of all the rows under the final solution column P are all **trues**, then it is a **tautology**.

Contradiction
A proposition P is a **contradiction** if and only if P resolves to **false** for every possible truth value assignment of its variables. In other words, if the truth values of all the rows under the final solution column P are all **falses**, then it is a **contradiction**.

Exercise 1.4 *Which of the two above formulae is a tautology, and which is a contradiction?*

Example 1.4 *Show that $\neg p \to (\neg p \vee q)$ is a tautology.*

One simple way to determine if a proposition is a tautology or not is to construct a truth table for the formula.

p	q	$\neg p$	$\neg p \vee q$	$\neg p \to (\neg p \vee q)$
T	T	F	T	**T**
T	F	F	F	**T**
F	T	T	T	**T**
F	F	T	T	**T**

Because every truth values under $p \vee (p \to q)$ is true, therefore, $p \vee (p \to q)$ is a tautology.

Example 1.5 *Determine if $p \to (p \to q)$ is a tautology.*

Once again we will use a truth table:

p	q	$p \to q$	$p \to (p \to q)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

If $p = T$ and $q = F$, then $p \to (p \to q)$ resolves to a F. Therefore, $p \to (p \to q)$ is not a tautology.

Notice something peculiar we did for this example. We can clearly see from the truth table that $p \to (p \to q)$ is not a tautology because the rows are not all trues. However, in order to definitively show that a proposition is not a tautology, we have to give a specific instance of when $p \to (p \to q)$ does not resolve to true, contradicting the definition of a tautology. In this case we specifically say that when $p = T$ and $q = F$, the proposition $p \to (p \to q)$ resolves to **false**, and is therefore not a tautology as it break the definition of a tautology. To show a proposition is not a tautology, you must provide a **counter-example**.

Counter-example
A specific example that refutes or disproves a proposition or theory.

Similarly, to show that a compound proposition Q is not a contradiction, you must also provide a counter-example in which for a specific assignment of truth values of its variables Q will resolve to true.

Exercise 1.5 *Determine if $p \land q \land r$ is a contradiction*

Exercise 1.6 *Given a compound proposition A, is the formula $(p \land \neg p) \to A$ a tautology or a contradiction?*

Just from knowing if a proposition is a **tautology** or a **contradiction**, we can resolve rather ambiguous statements to get a better idea of how they behave.

Example 1.6 *Given a compound proposition A that is a tautology and a proposition p, what do we know about $p \vee A$ and $p \rightarrow A$?*

This is a rather odd question, but let's try making truth table. Since we know that A is a **tautology**, we know that it must <u>always resolve to a true</u>. p is any proposition, so it can be either true or false.

p	A	$p \vee A$	$p \rightarrow A$
T	T	T	T
F	T	T	T

We can see from our truth table that both $p \vee A$ and $p \rightarrow A$ <u>always resolves to true</u>, therefore they are also **tautologies**.

We can also come to the same conclusion without using a truth table if we take a closer look at how **disjunction** and **implication** behaves:

- **Disjunction** is looking for any presence of a **true** in either of its propositions. Since A is always guaranteed to be **true** by definition of a tautology, that means the condition for disjunction to be **true** is always satisfied, therefore $p \vee A$ will <u>always resolve to true</u>, a **tautology**.
- **Implication** has two scenarios: 1) If the hypothesis is **false**, then the implication will always resolve to **true**, ignoring the conclusion, therefore we only need to check the second scenario. 2) If the hypothesis is **true**, then the implication will take the behavior of the conclusion. Since the conclusion A is always true by definition of a tautology, we are guaranteed that the implication $p \rightarrow A$ will <u>always resolve to true</u>, a **tautology**.

This type of reasoning without a truth table may seem a abstract, but it is much more efficient than drawing out a truth table. A truth table is feasible for simple compound propositions, but as the propositions we are looking at become more and more complex, a truth table become less and less feasible. We will transition away from truth tables when we look at Laws of Propositional Logic in Section 1.5.

Exercise 1.7 *Given two compound propositions A and B, where A is a contradiction and B is a tautology, determine if the following compound propositions resolve to a tautology or a contradiction.*

(a) $A \wedge p$

(b) $B \vee p$

(c) $A \rightarrow B$

(d) $A \rightarrow p$

1.4 Translating Between English and Propositional Logic

English and other human languages can be really ambiguous. Consider the following sentence:

"I saw an elephant in my pajamas."

of which your response to this proposition might be:

"Why was the elephant wearing your pajamas?"

Of course, the actual meaning of the original sentence is that you saw an elephant while you, yourself, were wearing your own pajamas, but this was made ambiguous by the syntax of the English language.

Logic allows us to bypass this ambiguity. The beauty and elegance of logic is that once we translate human languages into logical propositions and its operators, there is no room for misinterpretation.

If you recall in the introduction, I compared learning logic to learning a new language. This will be your first experience in what I meant by this. To translate between English and logic, you will need to *interpret* and fully *understand* the meaning of the English sentence in order to translate it into logic and vice versa. This means we not only need to understand how to resolve the truth values of **logical operators**, we will need to understand what each **logical operator** actually means in context of a human language.

1.4.1 Meaning of the Logical Operators

In this section we will use examples to demonstrate the meaning of the logical operators. The examples will use the following atomic propositions:

- $h =$ it is hot outside.
- $c =$ it is cold outside.
- $s =$ it is summer.
- $w =$ it is winter.

Negation ($\neg$)

Negation is the easiest to identify and understand. This is used whenever we a the truth value of a proposition is reversed:

- $\neg h$: it is **not** hot outside.
- $\neg s$: it is **not** summer.

Notice that $\neg h$ does **not** have the same meaning as c, as "it is not hot outside" may mean "it is just warm outside", which is not the same meaning as "it is cold outside".

Conjunction ($\wedge$)

Conjunction is used when both propositions must be true at the same time.

- $s \wedge h =$ it is summer **and** it is hot outside.
- $w \wedge \neg c =$ it is winter **but** it is not cold outside.
- $s \wedge \neg h =$ it is summer, **however** it is not hot outside.

Notice that "and", "but", and "however", despite being very different English words, have the same logical meaning of both propositions being true at the same time. *"It is winter* ***but*** *it is not cold outside"* means that it is winter **and** it is not cold outside at the same time, thus a **conjunction**.

Disjunction ($\lor$) and Exclusive-or ($\oplus$)

Disjunction is used when the entire compound proposition will be **true** under the condition that any one of its atomic parts are true. Notice that it would still make sense when both propositions are true at the same time.

- $s \lor c =$ it is summer or it is cold outside.
- $w \lor \neg w =$ it is winter or it isn't.

For the first statement, it can be a cold summer day and $s \lor c$ will still hold true. For the second statement, while it cannot be both winter and not winter at the same time, this sentence holds true regardless what season we are in. If we are in winter, then we get $T \lor F \equiv T$; if it is any other season, then we still get $F \lor T \equiv T$. We are reading the sentence in its entirety and not just its atomic parts.

When we use *"or"* in daily life we often do not mean the logical **disjunction** but **exclusive-or**. Consider the following:

"Is it hot or cold outside?"

Our initial translation to logic is likely to be:

$$h \lor c$$

Strictly speaking for this translation of $h \lor c$ choosing both is a valid option since $T \lor T \equiv T$. What is actually meant is:

"Is it hot or cold outside (it cannot be both at the same time)?"

since we are restricted to one and only one choice, our translation is now:

$$h \oplus c$$

In English we implicitly understand that it cannot be both hot and cold outside at the same time. We use the **exclusive-or** operator when one and only one proposition can be true.

- $h \oplus c =$ it is either hot or it is cold outside, but not both at the same time.
- $s \oplus w =$ it is either summer or it is winter, but not both at the same time.

Example 1.7 *Determine if the following sentence should use disjunction ($\lor$) or exclusive-or ($\oplus$):*

1. **"Should we go play basketball or volleyball today?"**

   A sports aficionado might respond with "Why not both? Let's play basketball in the morning and volleyball in the afternoon." Since doing both options make sense, this is a **disjunction** ($\lor$).

2. **"Should we go play basketball or volleyball right now?"**

   Since we can only play one sport at a time, we must choose one over the other, therefore this is an **exclusive-or** ($\oplus$).

Implication ($\rightarrow$)

If we have $p \rightarrow q$, it means that q is <u>contingent</u> only when p is **true**.

- $w \rightarrow c =$ **If** it is winter, **then** it is cold outside.
- $s \rightarrow h =$ It is hot outside **if** it is summer.
- $h \rightarrow d =$ It is hot outside **only if** I would like a salad.
- $c \rightarrow p =$ I would like soup **when** it is cold outside.
- $s \rightarrow d =$ The summer season **implies** I would like salad.

It is easy to recognize when we would use implication, however you have to be **very** careful with the direction of the implication.

In general, the following are all proper translations of $p \rightarrow q$:

q if p	p only if q	q provided that p
q in the case of p	p results in q	q whenever p
q follows from p	p is sufficient for q	q is necessary for p

Exercise 1.8 *Translate the following sentences to logic:*

Let:

e: The number is even.
d: The number is divisible by 2.

1. "A number is even if it is divisible by 2."

2. "A number is even only if it is divisible by 2."

Equivalence ($\longleftrightarrow$)

Equivalence (or bi-implication or bi-conditional) is used when the two propositions have the exact same meaning as each other or when their truth values behave in the exact same way.

- $h \longleftrightarrow s =$ it is hot outside **if and only if** it is summer.
- $c \longleftrightarrow w =$ the outside being cold **is equivalent to** it being winter.
- $h \longleftrightarrow d =$ **if** it is hot outside then I would like salad **and vice versa** (hence the name "bi-conditional").

You will notice throughout the course we will use bi-conditional a lot when defining terms as it provides the precise logical meaning of equivalency between the terminology and its mathematical definition.

1.4.2 Translating English into Propositional Logic

Now that we know the meaning behind the logical operators, we can now proceed to translate full English propositional sentences into **propositional formulae**.

When we translate English into propositional logic we will do the following:

Steps for Translating English into Propositional Logic

1. Identify the **atomic propositions** and assign them **propositional variables**.
2. Identify the **logical operators** used in the sentence.
3. Determine the order in which the variables are connected (especially important for implication).

Example 1.8 *Translate "I would like salad if it is summer and it is hot outside" into propositional logic.*

First, we will need to identify the atomic propositions and assign them variables:

- $d =$ I would like salad.
- $h =$ it is hot outside.
- $s =$ it is summer.

As tempting as it is to assign a single variable to "it is summer and hot outside", this statement is a **compound** proposition, so we must break it down to its **atomic** parts.

Next we need to identify the logical operators. Here the keywords **"and"** and **"if"** are our keywords to recognize that we will use $\land$ and $\rightarrow$ in our formula.

Now we have to do a bit of interpreting. The sentence says that it being summer and hot outside is the condition for which I would like salad. Therefore, we will need to combine s and h together as $s \land h$ since both of these are the condition to the implication statement. Next, since $s \land h$ is the condition, the direction of our implication is from $s \land h$ to d, therefore:

$$(s \land h) \rightarrow d$$

Be very careful not to naively translate the English sentence left to right, which would result in the incorrect translation $d \rightarrow (s \land h)$.

Exercise 1.9 *Translate "when it snows or the roads freeze, the roads will be dangerous and the school will close" into propositional logic.*

1.4.3 Translating Propositional Logic into English

Translating logic into English is more straightforward, however it will require you to do a bit of interpretation to make your translation human-sounding.

Steps for Translating Propositional Logic into English

1. Substitute in the **atomic propositions** and the **logical operators**.
2. If possible, rewrite the sentence to sound more natural.

Example 1.9 *Translate the following propositional formula into English.*

$$\begin{aligned} h &= \text{I am hungry} \\ t &= \text{I am tired} \\ s &= \text{I am sleepy} \\ w &= \text{I need to get work done} \end{aligned}$$

$$h \wedge t \wedge s \wedge w$$

Here we can substitute in the atomic propositions and logical operators to get:

"I am hungry and I am tired and I am sleepy and I need to get work done."

However, who talks like this? Well, a four year-old might, or if someone is complaining and wants to emphasize their frustration. But we can make this sound more natural/"normal" by writing the following:

"I am hungry, tired, and sleepy, but I need to get work done."

Notice that we do not alter the logical meaning in any way with the second sentence.

Exercise 1.10 *Translate the following into English.*

$$\begin{aligned} i &= \text{I am from Indiana.} \\ p &= \text{I am a fan of the Indiana Pacers.} \\ f &= \text{I am a fan of Indy500.} \end{aligned}$$

$$i \wedge \neg p \wedge \neg f$$

1.5 Logical Equivalencies

When children begin to understand the meaning of negations, many people, myself included, would say silly things like *"I will not not eat the chocolate"* or *"I did not not finish my homework"* just to annoy their parents. This is perhaps due to the fascination of realizing that *"I will not not eat the chocolate"* has the exact same meaning as *"I will eat the chocolate"* due to the double negation.

And so now, as university students, we will revisit this fascination and ask the question:

When do two propositional statements have the same logical meaning?

to which the answer is:

When the two propositional statements always resolve to the same truth values.

or more formally:

> **Logical Equivalency**
> Given two compound propositions P and Q, P and Q are **logically equivalent** if and only if $P \longleftrightarrow Q$ is a tautology. We use the notation $P \equiv Q$ to denote this equivalence.

1.5.1 Determining Logical Equivalency using Truth Tables

We can test for **logical equivalency** by constructing a **truth table** for both propositions. If the columns for the two propositional statements have the exact same matching truth values for every row, then we know that they always resolve to the same truth value, thus they are **logically equivalent**.

Example 1.10 *Use a truth table to determine if $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.*

Let's construct the truth table that will resolve both formulas:

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	**F**	**F**
T	F	F	T	F	**T**	**T**
F	T	T	F	F	**T**	**T**
F	F	T	T	F	**T**	**T**

Because every row for the $\neg(p \wedge q)$ and $\neg p \vee \neg q$ columns are exactly the same, we can say; therefore $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are **logically equivalent**.

Example 1.11 *Use a truth table to determine if $p \rightarrow q$ and $q \rightarrow p$ are logically equivalent.*

Let's construct the truth table that will resolve both formulas:

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	**F**	**T**
F	T	**T**	**F**
F	F	T	T

Here we must provide a **counter-example** to show that they are **not logically equivalent**. If $p = T$ and $q = F$, then $p \rightarrow q$ resolves to F but $q \rightarrow p$ resolves to T, therefore $p \rightarrow q$ and $q \rightarrow p$ are not logically equivalent.
Note: $p = F$ and $q = T$ is another valid counter-example.

Exercise 1.11 *Use a truth table to determine if* $p \to q$ *and* $\neg p \vee q$ *are logically equivalent.*

Exercise 1.12 *Use a truth table to determine if* $\neg(p \vee q)$ *and* $\neg p \vee \neg q$ *are logically equivalent.*

1.5.2 Determining Logical Equivalency using Laws of Propositional Logic

Example 1.12 *Use a truth table to determine if* $(a \vee \neg a) \wedge (b \vee \neg b) \wedge \cdots \wedge (z \vee \neg z)$ *is logically equivalent to* T *(a tautology).*

Well, since it tells us to use a truth table, let's make one:

a	b	$\cdots$	z	$\neg a$	$\cdots$	$\neg z$	$\cdots$
T	T	$\cdots$	T	F	$\cdots$	F	$\cdots$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	
F	F	$\cdots$	F	T	$\cdots$	T	$\cdots$

With 26 unique variables, we'll need 2^{26} number of rows, which is more than 67 million rows. This is infeasible, even for a computer.

Is there a better strategy than using truth tables?

If you recognize $a \vee \neg a$ from the exercises you did in finding **tautologies**, you know that $a \vee \neg a$ will always resolve to true. By the same logic, $b \vee \neg b$ will also be always true. Therefore:

$$(a \vee \neg a) \wedge (b \vee \neg b) \wedge \cdots \wedge (z \vee \neg z) \equiv T \wedge T \wedge \cdots \wedge T$$

which now we can trivially see that it will resolve to T, therefore it is a **tautology**. This strategy allows us to find the answer in two steps rather than 67 million steps.

This approach of understanding that $a \vee \neg a \equiv T$ and $T \wedge T \equiv T$ is what we refer to as using **laws of propositional logic**. Below we give you the table of all the laws you need to use for this approach:

$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$	Commutative law
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative law
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive law
$p \wedge p \equiv p$ $p \vee p \equiv p$	Idempotence law
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity law
$p \wedge \mathbf{F} \equiv \mathbf{F}$ $p \vee \mathbf{T} \equiv \mathbf{T}$	Domination law
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's law
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation law
$\neg\neg p \equiv p$	Double negation
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption law
$p \rightarrow q \equiv \neg p \vee q$	Implication in terms of disjunction
$p \longleftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	Bi-implication in terms of implication

Do not try to memorize this table. Most of these laws you can intuit with a bit of reasoning and quick scratch work. The laws that are less intuitive that you should commit to memory are **distributive**, **DeMorgan's laws**, **implication in terms of disjunction**, and **bi-implication in terms of implication**.

Example 1.13 *Use the laws of propositional logic to show* $p \to (q \to r) \equiv q \to (p \to r)$.

$$
\begin{aligned}
p \to (q \to r) &\equiv p \to (\neg q \lor r) && \text{Implication in terms of disjunction} \\
&\equiv \neg p \lor (\neg q \lor r) && \text{Implication in terms of disjunction} \\
&\equiv (\neg p \lor \neg q) \lor r && \text{Associative law} \\
&\equiv (\neg q \lor \neg p) \lor r && \text{Commutative law} \\
&\equiv \neg q \lor (\neg p \lor r) && \text{Associative law} \\
&\equiv \neg q \lor (p \to r) && \text{Implication in terms of disjunction} \\
&\equiv q \to (p \to r) && \text{Implication in terms of disjunction}
\end{aligned}
$$

Here is a general guideline to using **laws of propositional logic**:

- Whenever you see a T or a F, there is a law that can resolve it (e.g., $p \lor T$, $F \land q$).
- Whenever you see a logical operator between two of the same variable (or negations of itself), there is a law that can resolve it (e.g., $p \lor p$, $\neg q \land q$).
- There are more laws that operate on $\lor$ and $\land$, therefore it is generally a good idea to use $p \to q \equiv \neg p \lor q$.
- Be aware that $\lor$ can always be transformed back into a $\to$.
- Be on a lookout for opportunities to use distributive and De Morgan's law.
- Start from the more complicated side.
- Keep your goal in mind throughout the proof.
- Stick to rules provided in the course.
- Have a plan.

Exercise 1.13 *Use the laws of propositional logic to show* $p \to (q \to r) \equiv (p \land q) \to r$.

Exercise 1.14 *Use the laws of propositional logic to show* $(p \to r) \land (q \to r) \equiv (p \lor q) \to r$.

Exercise 1.15 *Use the laws of propositional logic to show* $p \to (q \to p)$ *is a tautology.*

Exercise 1.16 *Use the laws of propositional logic to show* $(p \land (p \to q)) \land \neg q$ *is a contradiction.*

Exercise 1.17 *Use the laws of propositional logic to show* $(p \to q) \longleftrightarrow (\neg q \to \neg p)$ *is a tautology.*

1.6 Propositional Arguments

At this point in the course, you have learned the basics of **propositional logic**, the different types of **logical operators**, and how to determine **logical equivalencies**. This sets the groundwork for us to discuss one of the more applicable aspects in logic: **arguments**. We often say or hear the phrase, *"your argument is flawed"*. In this section we will learn to say *"your argument is flawed and I can prove it mathematically"*.

To start, let's start with a fun little story. Greek philosopher *Plato* was once applauded by his colleagues for his definition of a man:

A man is a featherless biped.

of which *Diogenes*, *Plato's* rival and aptly known as *Diogenes the Cynic*, plucked all the feathers from a chicken, burst into *Plato's* school with the poor bird in hand, and boldly proclaimed:

"BEHOLD! Here is Plato's man!"[5]

Diogenes did two clever things here. First, he provided a **counter-example** to *Plato's* definition of a man. The nude chicken is indeed *a featherless biped* but evidently not a man, therefore proving *Plato's* definition to be flawed. Second, *Diogenes* had formally shown that *Plato's* **argument** was **invalid**.

1.6.1 What is a Logical Argument?

Let us first define what an argument is:

> **Propositional Argument**
> A **propositional argument** is comprised of a set of **hypotheses** followed by a **conclusion**. The **hypothesis** are propositions that are held to be **true**, and the **conclusion** is the proposition claimed to be logically derived from the hypothesis.

Plato's (flawed) definition of a human is actually an argument he constructed as such:

	All humans are bipedal.
	All humans are featherless.
$\therefore$	All bipedal, featherless animals are human.

The logic used by *Plato* and *Diogenes* is actually called **predicate logic**, which we will learn in **Unit 2**. So for now, we'll simplify *Plato's* argument as:

	I am bipedal.	$\leftarrow$ Hypothesis 1
	I am featherless.	$\leftarrow$ Hypothesis 2
	If I am bipedal and featherless, then I am a human	$\leftarrow$ Hypothesis 3
$\therefore$	I am a human.	$\leftarrow$ Conclusion

[5] Disclaimer: this is a dramatic retelling of the story.

Here, *"I am bipedal"*, *"I am featherless"*, and *11If I am bipedal and featherless, then I am a human"* are the **hypothesis**, which are the propositions that *Plato* and we all agree to be true. *"I am a human"* is the **conclusion**, which is the proposition that *Plato* is claiming to be true.

It is important to note here that another way to define an **argument** is the construction of a logical implication from hypothesis to the conclusion. The statement:

"I am bipedal and I am featherless, therefore I am a human."

is logically equivalent to:

"If I am bipedal and featherless, then I am a human."

Formally, a logical argument is claiming that a propositional conclusion is implied by a set of conjuntive hypotheses is a tautology. A hypothesis is true by definition, thereofre we only have to check if the conclusion is true under the condition when all the hypotheses are also true. Rather than a dictionary definition, we can define a more succinct version of an argument in formal logic:

Propositional Argument (in Formal Logic)
Let $H_1, \cdots, H_n$ be the set of n hypothesis. Let C be the conclusion. An argument is thus defined as $(H_1 \wedge \cdots \wedge H_n) \rightarrow C$; an argument can also be written in vertical form:

$$\begin{array}{rl} & H_1 \\ & \vdots \\ & H_n \\ \hline \therefore & C \end{array}$$

1.6.2 Translating from English into Logical Arguments

Much similar to translating English to propositional logic, you will have to identify the **atomic propositions**, the **logical operators**, and construct your **propositional formulae** accordingly; except, now you also have to identify which formulae are **hypothesis** and which are **conclusions**.

Example 1.14 *Translate the following argument into logic.*

If I am lactose intolerant and eat dairy, then I will get a stomachache. If I have a stomachache, I will not be able to finish my homework. I ate dairy even though I am lactose intolerant, therefore I will not be able to finish my homework.

This is quite a long argument, but as long as we methodically identify the **atomic propositions** and **logical operators**, we'll be just fine.

$$\begin{aligned} l &= \text{I am lactose intolerant.} \\ e &= \text{I eat dairy.} \\ s &= \text{I get a stomachache.} \\ f &= \text{I finish my homework.} \end{aligned}$$

Notice two tricky steps we did here: 1) we did not assign a variable to the statement *"I will not be able to finish my homework"* because this proposition still has a negation it, therefore we need to break it down further into its **atomic proposition** of just *"I will finish my homework."* 2) Even though we don't have a straight up translation for p *even though* q", we can take a moment to understand that in this sentence we are both lactose intolerant and eating dairy at the same time, therefore this is a **conjunction**.

Now, let's use the **atomic propositions** and reading the **logical operators** used, translate each sentence into a **propositional formulae**:

$$
\begin{array}{l}
(l \wedge e) \rightarrow s \\
s \rightarrow \neg f \\
d \wedge l \\
\neg f
\end{array}
$$

And finally, all we need to do now is to identify which sentence is the **conclusion** of the argument. Here, we can clearly see that the final sentence has *"therefore"* beginning it, thus the last sentence must be the **conclusion**. We can now complete the translation by writing the formulae in argument form:

$$
\begin{array}{rl}
 & (l \wedge e) \rightarrow s \\
 & s \rightarrow \neg f \\
 & d \wedge l \\
\hline
\therefore & \neg f
\end{array}
$$

You can, of course, translate the formulae and identify the conclusion in one step to save you some time, if you wish so.

Exercise 1.18 *Translate the following argument into logic.*

I am a chicken. If I am a chicken, then I am bipedal, have feathers, and have a beak. Being bipedal and featherless is the necessary and sufficient condition to be a human. Therefore I am a human with a beak.

What are the atomic propositions? Assign variables to them.

Which proposition is the conclusion? Which are the hypothesis?

Which logical operators is each sentence using? Translate and write in argument form:

1.6.3 Translating from Propositional Arguments into English

The process is very much similar to translating propositional statements into English, except we now have to also include the conclusion in our translation.

Exercise 1.19 *Translate the following propositional argument into English.*

$e =$ The elephant is wearing my pajamas.
$p =$ I am wearing my pajamas.
$d =$ I am dreaming.
$b =$ My pajamas are too big for me.

$$\begin{array}{ll} & p \oplus e \\ & e \\ & \neg p \rightarrow d \\ & e \rightarrow b \\ \hline \therefore & d \vee b \end{array}$$

1.7 Argument Validity

At this point we've seen some pretty weird arguments and absurd conclusions. And yet, some of them sound like they make logical sense. How do we determine if an argument is valid or not? Here, we define what **argument validity** means.

> **Argument Validity**
> An **argument** is considered to be **valid** if and only if the **conclusion** under the condition for when all **hypotheses** are also true. More formally, given an argument with the set of hypotheses $H_1, \cdots, H_n$ and the conclusion C, the argument is valid if and only if the following is a **tautology**:
>
> $$(H_1 \land \cdots \land H_n) \to C$$

For very similar reasons as when we looked at **logical equivalence**, we will look at two methods to determining the validity of an argument: 1) using a **truth table** and 2) using **rules of inference**.

1.7.1 Determining Validity using a Truth Table

If we construct a truth table resolving all the **hypotheses** and the **conclusion**, we much check that the **conclusion** is true in all the rows where the all **hypotheses** are also true. The steps to do this are:

1. Construct a truth table resolving all the **hypotheses** and the **conclusion** (one column for each hypothesis and the conclusion).
2. Identify the columns that represent the **hypotheses** and the **conclusion**.
3. Identify the **critical rows**.
4. Check to see if the **conclusion** is true in all the **critical rows**:
   - If the **conclusion** is true in all the **critical rows**, then the argument is **valid**.
   - If any of the **critical rows** contain a conclusion resolved to **false**, then disprove the argument by using a **counter-example**.

> **Critical Rows**
> A **critical row** is a row in a truth table where all the **hypotheses** are true. There can be multiple **critical rows** in a single truth table. The **critical rows** are not determined by the **conclusion**.

Let's take this step by step with an example:

Example 1.15 *Determine if the argument* $\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$ *is valid using a truth table.*

Let's first construct a truth table that will resolve p, q, and $p \to q$:

hypothesis 2	Conclusion	hypothesis 1
p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Now we have to identify the **critical rows**. These are the rows where all the hypotheses (in this case, $p \to q$ and p) are true at the same time.

hypothesis 2 p	Conclusion q	hypothesis 1 $p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

in this case, we only have one **critical row**.

Because the **conclusion** q is true in all the **critical rows**, the argument is **valid**.

Let's take a look at a more complicated example:

Example 1.16 *Determine if the argument*

$$\begin{array}{c} p \to (q \vee \neg r) \\ q \to (p \wedge r) \\ \hline \therefore p \to r \end{array}$$

is valid using a truth table.

Once again, let's construct a truth table resolving the **hypotheses** and **conclusion**. Afterwards, identify the **critical rows**.

p	q	r	$\neg r$	$q \vee \neg r$	$p \wedge r$	hypothesis 1 $p \to (q \vee \neg r)$	hypothesis 2 $q \to (p \wedge r)$	Conclusion $p \to r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	F
T	F	T	F	F	T	F	T	T
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	T
F	T	F	T	T	F	T	F	T
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

Here, we see that there are 4 **critical rows**. When we check the truth value of the **conclusion** of each **critical row**, we see that in row 4, the conclusion has resolved to a false, meaning the argument is **invalid**. To show that an argument is invalid, you must provide a **counter-example**:

If $p = T, q = F, r = F$, then the hypotheses $p \to (q \vee \neg r)$ and $q \to (p \wedge r)$ remain consistent (they are still true), however the conclusion $p \to r$ is false. Therefore, the argument is **invalid**.

Notice that our counter-example is written as another argument. While this may seem like an absurd thing to do and is confusing, this avoids all ambiguity and allows us to definitively disprove the original argument.

Exercise 1.20 *Determine if the argument*

$$\begin{array}{l} p \to q \\ q \to r \\ p \\ \hline \therefore r \end{array}$$

is valid using a truth table.

Which formulae are the hypotheses? Which is the conclusion?

Construct a truth table resolving each of the hypotheses and the conclusion. How many rows do you need for this truth table?

Which rows are the critical rows? What is the truth value of the conclusion of each critical row? Are they all trues? Are there any falses?

Is the argument valid or invalid? Why or why not?

Exercise 1.21 *Determine if the argument* $\begin{array}{c} p \vee r \\ p \to q \\ r \to q \\ \hline \therefore p \end{array}$ *is valid using a truth table.*

Example 1.17 *Translate the following argument into logic and determine if it is valid using a truth table. "If I study business, then I will make a lot of money. If I study informatics, then I will enjoy my career. If I make a lot of money or enjoy my career, then I will not be disappointed. Therefore, if I am disappointed, then I neither studied business nor studied informatics."*

We first identify the atomic propositions:

$$b = \text{I study business.}$$
$$m = \text{I will make a lot of money.}$$
$$i = \text{I study informatics.}$$
$$e = \text{I will enjoy my career.}$$
$$d = \text{I will be disappointed.}$$

Now we construct a truth table and identify the critical rows to determine if the argument is valid.

b	m	i	e	d	$b \to m$	$i \to e$	$\neg d$	$m \vee e$	$(m \vee e) \to \neg d$	$\neg b$	$\neg i$	$\neg b \wedge \neg i$	$d \to (\neg b \wedge \neg i)$
T	T	T	T	T	T	T	F	T	F	F	F	F	F
T	T	T	T	F	T	T	T	T	T	F	F	F	T
T	T	T	F	T	T	F	F	T	F	F	F	F	F
T	T	T	F	F	T	F	T	T	T	F	F	F	T
T	T	F	T	T	T	T	F	T	F	F	T	F	F
T	T	F	T	F	T	T	T	T	T	F	T	F	T
T	T	F	F	T	T	T	F	T	F	F	T	F	F
T	T	F	F	F	T	T	T	T	T	F	T	F	T
T	F	T	T	T	F	T	F	T	F	F	F	F	F
T	F	T	T	F	F	T	T	T	T	F	F	F	T
T	F	T	F	T	F	F	F	F	T	F	F	F	F
T	F	T	F	F	F	F	T	F	T	F	F	F	T
T	F	F	T	T	F	T	F	T	F	F	T	F	F
T	F	F	T	F	F	T	T	T	T	F	T	F	T
T	F	F	F	T	F	T	F	F	T	F	T	F	F
T	F	F	F	F	F	T	T	F	T	F	T	F	T
F	T	T	T	T	T	T	F	T	F	T	F	F	F
F	T	T	T	F	T	T	T	T	T	T	F	F	T
F	T	T	F	T	T	F	F	T	F	T	F	F	F
F	T	T	F	F	T	F	T	T	T	T	F	F	T
F	T	F	T	T	T	T	F	T	F	T	T	T	T
F	T	F	T	F	T	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	F	T	F	T	T	T	T
F	T	F	F	F	T	T	T	T	T	T	T	T	T
F	F	T	T	T	T	T	F	T	F	T	F	F	F
F	F	T	T	F	T	T	T	T	T	T	F	F	T
F	F	T	F	T	T	F	F	F	T	T	F	F	F
F	F	T	F	F	T	F	T	F	T	T	F	F	T
F	F	F	T	T	T	T	F	T	F	T	T	T	T
F	F	F	T	F	T	T	T	T	T	T	T	T	T
F	F	F	F	T	T	T	F	F	T	T	T	T	T
F	F	F	F	F	T	T	T	F	T	T	T	T	T

1.7.2 Determining Validity using Rules of Inference

The taxonomical categorization of a domestic chicken is:

Kingdom	=	Animalia
Phylum	=	Chordata
Class	=	Aves
Order	=	Galliformes
Family	=	Phasianidae
Genus	=	Gallus

therefore we can construct the following argument:

If I am a domestic chicken, then I am in the Gallus genus.
If I am in the Gallus genus, then I am in the Phasianidae family.
If I am in the Phasianidae family, then I am in the Galliformes order.
If I am in the Galliformes order, then I am in the Aves class.
If I am in the Aves class, then I am in the Chordata phylum.
If I am in the Chordata phylum, then I am in the Animalia kingdom.
I am a domestic chicken

$\therefore$ I am in the Animalia kingdom.

Just from reading through the argument, we know that this argument is valid. However proving that the argument is valid is another story. We have 7 unique propositions, therefore we will need $2^7 = 128$ rows.

Who wants to construct a table with 128 rows? That's terribly labor intensive and unnecessary. There must be a better way, and there is.

When you read the argument, you can logically see why the argument is valid. The hypothesis starts us off with *"I am a chicken"*, and we know that *"if we are a domestic chicken, then we are in the Gallus genus"*, therefore we must be in the Gallus genus. We can then follow the implication hypotheses and eventually reach the conclusion of *"I am in the Animalia kingdom"*. This process is called **deduction**.

> **Deduction**
> A form of reasoning that infers conclusions from a set of hypotheses.

In this section we will use **rules of inference** to prove the validity of an argument.

> **Rules of Inference**
> A reasoning method in which starting with the **hypotheses** of an argument and using known **inference rules** we will infer new propositions until we reach the **conclusion** of the argument, thereby proving the validity of the argument.

There are also other formal proof systems similar to rules of inference such as *rules of inference*. However, regardless of the system, we are all trying to prove the validity of an argument through the use of logic. This is the same formal process in which mathematics, physics, and other positivist fields prove new theorems and generate new knowledge.

Let's consider the following argument:

$$\begin{array}{rl} & P \\ & Q \\ & R \\ \hline \therefore & (P \wedge Q) \wedge R \end{array}$$

For brevity, let's use a shorthand notation for writing arguments. We can also write this argument as:

$$P, Q, R \vdash (P \wedge Q) \wedge R$$

Remember, the hypotheses P, Q, and R are what is considered to be true for the argument. Since we know P and Q are individually true, can logically conclude that $P \wedge Q$ must also be true. And now that we know $P \wedge Q$ to also be definitely true, combined with the hypothesis of R being true, we can logically conclude that $(P \wedge Q) \wedge R$ must also be true, therefore proving our conclusion.

More formally in **rules of inference**, we will use the following tabular format to organize our proof:

Example 1.18 *Use rules of inference to prove* $P, Q, R \vdash (P \wedge Q) \wedge R$

1	P	
2	Q	
3	R	
4	$P \wedge Q$	*Conjunction*, 1, 2
5	$(P \wedge Q) \wedge R$	*Conjunction*, 3, 4

Rules of inference
A system of logic where a conclusion is proven through a series of inferential rules on a set of given hypothesis. This proof is structured as a table shown. The first n rows are the **hypotheses**, followed by m rows of inferences to reach the conclusion R.

1	H_1	
2	H_2	
$\vdots$	$\vdots$	
n	H_n	
$n+1$	H_{n+1}	R_{n+1}
$n+2$	H_{n+2}	R_{n+2}
$\vdots$	$\vdots$	$\vdots$
$n+m-1$	H_{n+m-1}	R_{n+m-1}
$n+m$	$C = H_{n+m}$	R_m

It is important to keep in mind that every new line we write in our rules of inference is a logical consequence of one or more of the previous lines, meaning every new line we write is an **inferred hypothesis** that we know to be absolutely true. This is how we produce new **hypotheses** to use to eventually infer the **conclusion** to be true.

We will now examine the different **inference rules** we can use to infer new hypotheses to reach our conclusion.

Conjunction

$\vdots$	$\vdots$	
i	P	
$\vdots$	$\vdots$	
j	Q	
$\vdots$	$\vdots$	
	$P \land Q$	Conjunction i, j

We've seen this in the previous example. If we have with the hypotheses P (on line i) and Q (on line j) to be true by definition, then we can infer that $P \land Q$ together must also be true by using lines i and j.

Simplification

$\vdots$	$\vdots$	
i	$P \land Q$	
$\vdots$	$\vdots$	
	P	Simplification i
	Q	Simplification i

This is the opposite of **conjunction**. If we know $P \land Q$ to be true, then we know that the individual propositions P and Q must be true by themselves separately.

In rules of inference, we can use **laws of propositional logic** as a step of inference as equivalency holds as much validity as an inference rule. A **law of propositional logic** we will often see in our proof is the **double negation** law:

Double Negation

$\vdots$	$\vdots$	
i	$\neg\neg P$	
$\vdots$	$\vdots$	
	P	Double Negation i

Here is our classic double negation that we used when we were younger. If any proposition is double negated, then the proposition by itself must be true.

The next rule, **addition**, requires a bit of a creative step in logic:

Addition

$$
\begin{array}{r|l}
\vdots & \vdots \\
i & P \\
\vdots & \vdots \\
 & P \vee Q \quad \text{Addition } i
\end{array}
$$

Where did the proposition Q come from? Let's see why this is the case. On line i we know that P is absolutely true. If P is absolutely true, what would $P \vee Q$ be? If we refer back to the truth table for $\vee$, we will see that if we have a true anywhere in a row, $P \vee Q$ will resolve to true. This is why we can infer $P \vee Q$ from just P on line i without needing to know anything about Q because $P \vee Q$ will always resolve to true regardless of the truth value of Q.

With the same reasoning, a fun way to think about **simplification is:**

Simplification (Fun Version)

$$
\begin{array}{r|l}
\vdots & \vdots \\
i & P \\
\vdots & \vdots \\
 & P \vee \text{<ANYTHING YOU WANT>} \quad \text{Addition } i
\end{array}
$$

The last introductory rule we'll introduce is the **reiteration** rule:

Reiteration Rule

$$
\begin{array}{r|l}
\vdots & \vdots \\
i & P \\
\vdots & \vdots \\
 & P \quad \text{Reit. } i
\end{array}
$$

This is also trivially simple since if we know P to be true on line i, we can refer back to line i any time and **reiterate** that P is true. While this might seem useless, it is helpful for very long and complex proofs.

Example 1.19 *Use rules of inference to prove* $P \wedge Q, R \vdash (Q \wedge R) \vee Z$.

1	$P \wedge Q$	
2	R	
3	P	*Simplification*, 1
4	Q	*Simplification*, 1
5	$Q \wedge R$	*Conjunction*, 2, 4
6	$(Q \wedge R) \vee Z$	*Addition*, 5

Using **simplification** we can get lines 3 and 4 from line 1. We can then combine Q and R using **conjunction** on line 5. Finally since we've inferred $Q \wedge R$ to be true on line 5, we can now use *addition* to get $(Q \wedge R) \vee Z$ on line 6.

Exercise 1.22 *Use rules of inference to prove* $P \wedge Q, R \wedge D \vdash P \wedge D$.

Exercise 1.23 *Use rules of inference to prove* $D \wedge E, P \wedge Q \vdash (E \vee \neg Q) \vee P$.

The next set of rules we are going to look at are rather famous rules of inference that date all the way back to ancient Greeks in the 8^{th} century BC. These are **Modus Ponens**, **Modus Tollens**, **Disjunctive Syllogism**, **Hypothetical Syllogism**, and **Resolution**.

Modus Ponens (M.P. Rule)

$$
\begin{array}{r|ll}
\vdots & \vdots & \\
i & P \to Q & \\
\vdots & \vdots & \\
j & P & \\
\vdots & \vdots & \\
 & Q & MP, i, j
\end{array}
$$

You've actually proven this rule previously. Recall the beginning of determining the validity of arguments using truth tables, we proved that the argument $p \to q, p \vdash q$ is valid:

Exercise 1.24 *Re-prove the validity of* $\begin{array}{l} P \to Q \\ P \\ \hline \therefore Q \end{array}$ *using a truth table.*

We can clearly see why given the hypotheses P and $P \to Q$, we can then logically infer Q to be definitively true. **Modus Ponens** is one of the foundations of reasoning for mathematicians and can be used in very powerful ways.

Exercise 1.25 *Use rules of inference to prove* $P, Q, (P \land Q) \to Z \vdash Z$*.*

The next rule is very similar to **Modus Ponens**:

Modus Tollens (M.T. Rule)

$$\begin{array}{r|ll} \vdots & \vdots & \\ i & P \to Q & \\ \vdots & \vdots & \\ j & \neg Q & \\ \vdots & \vdots & \\ & \neg P & MT,\, i, j \end{array} \qquad \begin{array}{r|ll} \vdots & \vdots & \\ i & P \to \neg Q & \\ \vdots & \vdots & \\ j & Q & \\ \vdots & \vdots & \\ & \neg P & MT,\, i, j \end{array}$$

Exercise 1.26 *Prove the validity of* $\begin{array}{l} P \to Q \\ \neg Q \\ \hline \therefore \neg P \end{array}$ *using the truth table below:*

P	Q	$P \to Q$	$\neg Q$	$\neg P$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Notice that for **Modus Tollens** we begin with the same hypothesis $P \to Q$, but this time our second hypothesis is $\neg Q$, the negation of the conclusion. From this we can infer that $\neg P$ must be true (P must be false).

Exercise 1.27 *Use rules of inference to prove* $\neg P \to R, Q \to \neg R, Q \vdash P$.

Disjunctive Syllogism involves the $\vee$ operation, as indicated by the **"disjunctive"** in **disjunctive syllogism**.

Disjunctive Syllogism (D.S. Rule)

$\vdots$	$\vdots$		
i	$P \vee Q$		
$\vdots$	$\vdots$		
j	$\neg P$		
$\vdots$	$\vdots$		
	Q	$DS,\, i, j$	

$\vdots$	$\vdots$		
i	$\neg P \vee Q$		
$\vdots$	$\vdots$		
j	P		
$\vdots$	$\vdots$		
	Q	$DS,\, i, j$	

Exercise 1.28 *Prove the validity of* $\begin{array}{l} P \vee Q \\ \neg Q \\ \hline \therefore P \end{array}$ *using the truth table below or with intuitive reasoning.*

P	Q	$P \vee Q$	$\neg Q$
T	T	T	F
T	F	T	T
F	T	T	F
F	F	F	T

Instead of using a truth table, we can spend a bit of time to try to intuit this rule. If we are given the hypothesis of $P \vee Q$ to be absolutely true, and we know that P is false due to the $\neg P$ hypothesis, then the only way $P \vee Q$ to remain consistently true is for Q to also be true. If Q is false, then both P and Q will be false, resolving $P \vee Q$ to be false, contradicting the hypothesis, which cannot happen.

Exercise 1.29 *Use rules of inference to prove* $(P \wedge Q) \vee R, \neg R \vdash P$.

Exercise 1.30 *Use rules of inference to prove* $\neg R \vee \neg(P \wedge Q), P, Q \vdash \neg R$.

Now armed with these new inference rules, let's give some examples a try.

Exercise 1.31 *Use rules of inference to prove* $P, Q, (P \wedge Q) \to (R \wedge S) \vdash R$.

Exercise 1.32 *Use rules of inference to prove* $P \wedge Q, \neg(R \vee \neg Q) \to \neg P \vdash R$.

The next two rules are **hypothetical syllogism** and **resolution**. While the rules themselves are memorizable, we would also like to explore if we can intuit why these rules work. The intuiting of these two rules require a bit more creativity, but I highly recommend you spend time thinking through them. Intuiting these will help you identify when they should be used more efficiently.

Hypothetical Syllogism (H.S. Rule)

$$\begin{array}{r|l} \vdots & \vdots \\ i & P \to Q \\ \vdots & \vdots \\ j & Q \to R \\ \vdots & \vdots \\ & P \to R \quad HS,\, i, j \end{array}$$

To intuit this, we'll have to use implication in a rather interesting way. Let' suppose that P is true. Even though P is not a hypothesis that we can use, we can still pretend that it is true and see what happens. Supposing P is true, then we know that Q must also be true with **modus ponens**, and now that we have Q to be true, we can use **modus ponens** to get R to be true. But remember, because we're only *pretending* P to be true, R is not something we can conclude, instead we *supposed* P to be true and ended with R. In other words, **if** we suppose P is true, **then** we can conclude R is true. Or more simply, we can conclude **if** P **then** R: $P \to R$.

Resolution

$$\begin{array}{r|l} \vdots & \vdots \\ i & P \lor Q \\ \vdots & \vdots \\ j & \neg P \lor R \\ \vdots & \vdots \\ & Q \lor R \qquad Resolution,\, i, j \end{array}$$

Let's see if we can use some pretending to intuit this again. We see that we have P and $\neg P$ in our two hypothesis. Let's first see what happens if P is true. If P is true, then we can use **disjunctive syllogism** to conclude R to be true. Remember, R is not true in actuality, but only under the pretense that P is true. Now let's suppose $\neg P$ is true (P is false). If $\neg P$ is true, then we can conclude that Q is true using **disjunctive syllogism** again. While this might seem like a meaningless exercise, if we combine these two statements together, we get: if P is true, then we know R is true, however if P is false, then it means that Q is true, so regardless of what the truth value of P is, we know that **either** Q is true, **or** R is true: $Q \lor R$.

Exercise 1.33 *Use rules of inference to prove* $P \wedge (Q \rightarrow \neg R), \neg R \rightarrow S \vdash Q \rightarrow S$

Exercise 1.34 *Use rules of inference to prove* $\neg S \vee \neg T, (\neg S \vee \neg P) \rightarrow R, \neg P \vee T \vdash R$

Exercise 1.35 *Use rules of inference to prove* $A \vee B, \neg A \vee C, \neg B \vee C \vdash C$